Constraints, gauge symmetries, and noncommutative gravity in two dimensions

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Abstract

After an introduction into the subject we show how one constructs a canonical formalism in space-time noncommutative theories which allows to define the notion of first-class constraints and to analyse gauge symmetries. We use this formalism to perform a noncommutative deformation of two-dimensional string gravity (also known as Witten black hole).

Dedicated to Yu. V. Novozhilov on the occasion of his 80th birthday

1 Introduction

Over the past decade considerable progress has been achieved in noncommutative field theories [1]. These theories are defined on a manifold whose coordinates do not commute. There are two essentially equivalent ways to describe noncommutative coordinates. One either introduces operators instead of numbers, or defines a new product of functions on the manifold. Here we shall use the latter approach.

Noncommutativity is not a purely theoretical invention. Noncommutative coordinates is a feature of many physical systems. As an example one may consider electrons in an external magnetic field. If one then restricts the electrons to several lowest Landau level, one gets second class constraints. Dirac brackets of the coordinates are then nonzero. This situations is realized in the Quantum Hall Effect. Another important example comes from string theory. It has been demonstrated that coordinates of the end points of open string

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do not commute. Consequently, field theories on Dirichlet branes are noncommutative field theories. One can argue by using very general arguments [2] that already classical gravity implies noncommutativity of coordinates at short distances.

Let us now define the star product of functions which will replace usual point-wise product. Consider a space-time manifold $\mathcal{M}$ of dimension D. The Moyal star product of functions on $\mathcal{M}$ reads

$$f \star g = f(x) \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x)\,. \tag{1}$$

θ is a constant antisymmetric matrix. This product is associative, $(f \star g) \star h = f \star (g \star h)$. In this form the star product has to be applied to plane waves and then extended to all (square integrable) functions by means of the Fourier series. Obviously,

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}\,. \tag{2}$$

We impose no restrictions on θ, i.e. we allow for the space-time noncommutativity.

The Moyal product is closed,

$$\int_{\mathcal{M}} d^D x f \star g = \int_{\mathcal{M}} d^D x f \times g \tag{3}$$

(where $\times$ denotes usual commutative product), it respects the Leibniz rule

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g + f \star (\partial_\mu g), \tag{4}$$

and allows to make cyclic permutations under the integral

$$\int_{\mathcal{M}} d^D x f \star g \star h = \int_{\mathcal{M}} d^D x h \star f \star g\,. \tag{5}$$

The product (1) is not the only possible choice of an associative noncommutative product. The right hand side of (2) can depend, in principle, on the coordinates.

To construct a noncommutative counterpart of given commutative field theory on has to replace all point-wise products by the star products. The result is, of course, not unique. There are some natural restrictions on noncommutative deformations of field theories. For example, one usually requires that number of gauge symmetries is preserved by the deformation.

Among all noncommutative field theories the theories with space-time noncommutativity have a somewhat lower standing since it is believed that they cannot be properly quantised because of the problems with causality and unitarity (see, e.g., [3]). Such problems occur due to the time-nonlocality of these

theories caused by the presence of an infinite number of temporal derivatives in the Moyal star product. However, it has been shown later, that unitarity can be restored [4][1] (see also [6]) in space-time noncommutative theories and that the path integral quantisation can be performed [7]. This progress suggests that space-time noncommutative theories may be incorporated in general formalism of canonical quantisation [8]. Indeed, a canonical approach has been suggested in [9].

Apart from quantisation, there is another context in which canonical approach is very useful. This is the canonical analysis of constraints and corresponding gauge symmetries [8]. The problem of symmetries becomes extremely complicated in noncommutative theories. Already at the level of global symmetries one see phenomena which never appear in the commutative theories. For example, the energy-momentum tensor in translation-invariant noncommutative theories is not locally conserved (cf. pedagogical comments in [10]). At the same time all-order renormalizable noncommutative ϕ^4 theory is *not* translation-invariant [11]. A Lorentz-invariant interpretation of noncommutative space-time leads to a twisted Poincare symmetry [12]. It is unclear how (and if) this global symmetry can be related to local diffeomorphism transformations analysed, e.g., in [13]. Proper deformation of gauge symmetries of generic two-dimensional dilaton gravities remains on open problem (see below). Solving (some of) the problems related to gauge symmetries in noncommutative field theories by the canonical methods is the main motivation for this work.

We start our analysis from the very beginning, i.e. with a definition of the canonical bracket. Our approach is based on two main ideas. First of all, we separate implicit time derivatives (which are contained in the Moyal star), and explicit ones (which survive in the commutative limit). Only explicit derivatives define the canonical structure. As a consequence, the constraints and the hamiltonian become non-local in time. Therefore, the notion of same-time canonical brackets becomes meaningless. We simply postulate a bracket between canonical variables taken at different points of space ($\mathbf{x}$ and $\mathbf{x}'$) and of time (t and t'):

$$\{q_a(\mathbf{x},t), p^b(\mathbf{x}',t')\} = \delta_a^b \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{6}$$

This bracket is somewhat similar to the one appearing in the Ostrogradski formalism for theories with higher order time derivatives (see, e.g., [14] for applications to field theories and [9] for the use in space-time noncommutative theories), but there are important differences (a more detailed comparison is postponed until sec. 3).

[1] One has to note that the approach based on time-ordered perturbation theory has some internal difficulties [5].

Of course, the proposed formalism means a departure from the standard canonical procedure. Nevertheless, we are able to demonstrate that the new bracket satisfies such fundamental requirements as antisymmetry and the Jacobi identities. These brackets generate equations of motion. Moreover, one can define the notion of first-class constraints with respect to the new bracket and show that these constraints generate gauge symmetries of the action. We shall derive an explicit form of the symmetry transformation and see that they look very similar to the commutative case (the only difference, in fact, is the modified bracket and the star product everywhere). We stress that our bracket will be used here to analyse gauge symmetries of classical systems only. It is not clear whether such a bracket is useful for quantisation.

The main application of the canonical formalism proposed here is noncommutative gravity theories in two dimensions. Let us consider the commutative case first (see review [15] where one can also find a more extensive literature survey). Since the Einstein-Hilbert Lagrangian density in two dimensions is a total derivative, one has to introduce a scalar field ϕ (called dilaton) so that the action reads:

$$S = \int d^2x \sqrt{-g} \left[\frac{R}{2}\phi - \frac{U(\phi)}{2}(\nabla\phi)^2 + V(\phi) \right] . \tag{7}$$

This action is general enough to describe many important gravity theories in two dimensions. For example, the choice

$$V(\phi) = \Lambda\phi , \qquad U(\phi) = 0 \tag{8}$$

yields the Jackiw-Teitelboim (JT) model [16][2]. Spherically symmetric reduction of the Einstein theory in D dimensions leads to the dilaton gravity action in two dimensions with the potentials:

$$V(\phi) \propto \phi^{\frac{D-4}{D-2}} , \qquad U(\phi) \propto \frac{1}{\phi} . \tag{9}$$

The low energy limit of string theory [18] will be of particular importance for the present work. It is described by the potentials:

$$V(\phi) = -2\lambda^2\phi, \qquad U(\phi) = -\frac{1}{\phi} . \tag{10}$$

This model is also called the Witten black hole.

By a dilaton dependent conformal transformation $g_{\mu\nu} = e^{-2\rho}\tilde{g}_{\mu\nu}$ with

$$\rho = -\frac{1}{2}\int^{\phi} U(Y)dY \tag{11}$$

[2] The equations of motion for this model were first studied in [17].

one obtains an action for the metric $\tilde{g}$ again in the form (7) but with the potentials

$$\tilde{U} = 0\,, \qquad \tilde{V} = V\exp(-2\rho)\,. \tag{12}$$

For the string gravity (10) the potential

$$\tilde{V} = -2\lambda^2 \tag{13}$$

is a constant. Note, that the transformation $g \to \tilde{g}$ may be singular, so that conformally related theories describe, in general, globally inequivalent geometries. However, this conformal transformation may be very useful as it simplifies the local dynamics considerably.

The action (7) can be rewritten in the first order form:

$$S = \int \left[\phi_a De^a + \phi d\omega + \epsilon \left(\frac{\phi_a \phi^a}{2} U(\phi) + V(\phi) \right) \right] , \tag{14}$$

where we have used the Cartan notations, $e^a = e^a_\mu dx^\mu$ is the zweibein one-form, $a = 0, 1$ is the Lorentz index, $\omega = \omega_\mu dx^\mu$ is the spin-connection one-form (usual spin-connection is $\omega_\mu \varepsilon^{ab}$, with ε^{ab} being the Levi-Civita symbol). ϵ is the volume two-form. $De^a = de^a + \varepsilon^a{}_b\, \omega \wedge e^b$ is the torsion two-form. To prove the equivalence [19] one has to exclude auxiliary fields ϕ_a and the torsion part of ω by means of algebraic equations of motion. The rest then depends on e^a only through the metric $g_{\mu\nu} = e^a_\mu e_{\nu a}$ and is indeed equivalent to (7). The proof of quantum equivalence [20] is more tricky.

Commutative dilaton gravities in two dimensions are being successfully used to get an insight into such complicated problems as gravitational collapse, information paradox, and quantisation of gravity. In the noncommutative case only the JT model was treated in some detail in classical [21] and quantum [22] regimes. We also like to mention an alternative approach [23] to noncommutative geometry in two dimensions which does not use any particular action.

In this paper we construct another two-dimensional noncommutative dilaton gravity which is a deformation of conformally transformed string gravity and analyse its gauge symmetries by using the canonical analysis suggested below.

2 Canonical bracket

The phase space on $\mathcal{M}$ consists of the variables r_j which can be subdivided into canonical pairs q, p and other variables α which do not have canonical partners

(these will play the role of Lagrange multipliers or of gauge parameters). We define a bracket (r_j, r_k) to be ± 1 on the canonical pairs,

$$(q_a, p^b) = -(p^b, q_a) = \delta_a^b \tag{15}$$

and zero otherwise (e.g., $(\alpha, p) = (p^a, p^b) = 0$). With this definition the bracket (6) reads: $\{r_i(x), r_j(x')\} = (r_i, r_j)\delta(x - x')$. Note, that we are not going to use brackets between two local expressions (see discussion below).

Now we can define canonical brackets between star-local functionals on the phase space. We define the space of star-local *expressions* as a suitable closure of the space of free polynomials of the phase space variables r_j and their derivatives evaluated with the Moyal star. Such expressions integrated over $\mathcal{M}$ we call star-local *functionals*.

Locality plays no important role here, since after the closure one can arrive at expressions with arbitrary number of explicit derivatives (besides the ones present implicitly through the Moyal star). It is important, that all expressions can be approximated with only one type of the product (namely, the Moyal one), and no mixed expressions with star and ordinary products appear. One also has to define what does "suitable closure" actually mean, i.e. to fix a topology on the space of the functionals. This question is related to the restrictions which one imposes on the phase space variables. For example, the bracket of two admissible functionals (see (17) below) should be again an admissible functional. This implies that all integrands are well-defined and all integrals are convergent. Stronger restrictions on the phase space variables mean weaker restrictions on the functionals, and vice versa. Such an analysis cannot be done without saying some words about $\mathcal{M}$ (or about its' compactness, at least)[3]. We shall not attempt to do this analysis here (postponing it to a future work). All statements made below are true at least for $r \in C^\infty$ and for polynomial functionals (no closure at all).

Obviously, it is enough to define the bracket on monomial functionals and extend it to the whole space by the linearity. Generically, two such monomial functionals read:

$$R = \int d^D x\, \partial_{\kappa_1} r_1 \star \partial_{\kappa_2} r_2 \star \ldots \partial_{\kappa_n} r_n\,, \qquad \tilde{R} = \int d^D x\, \partial_{\tilde{\kappa}_1} \tilde{r}_1 \star \partial_{\tilde{\kappa}_2} \tilde{r}_2 \star \ldots \partial_{\tilde{\kappa}_m} \tilde{r}_m \tag{16}$$

κ_j is a multi-index, ∂_{κ_j} is a differential operator of order $|\kappa_j|$. The (modified)

[3]Some restrictions on $\mathcal{M}$ follow already from the existence of the Moyal product, which requires existence of a global coordinate system at least in the noncommutative directions.

canonical bracket of two monomials is defined by the equation

$$\{R,\tilde{R}\} = \sum_{i,j} \int d^D x\, \partial_{\kappa_j} \left(\partial_{\kappa_{j+1}} r_{j+1} \star \dots \partial_{\kappa_{j-1}} r_{j-1} \right) (r_j, \tilde{r}_i) \\ \star \partial_{\tilde{\kappa}_i} \left(\partial_{\tilde{\kappa}_{i+1}} \tilde{r}_{i+1} \star \dots \partial_{\tilde{\kappa}_{i-1}} \tilde{r}_{i-1} \right) (-1)^{|\kappa_j| + |\tilde{\kappa}_i|} \,. \tag{17}$$

In other words, to calculate the bracket between two monomials one has to (i) take all pairs r_j, $\tilde{r}_i$; (ii) use cyclic permutations under the integrals to move r_j to the last place, and $\tilde{r}_i$ – to the first; (iii) integrate by parts to remove derivatives from r_j and $\tilde{r}_i$; (iv) delete r_j and $\tilde{r}_i$, put the integrands one after the other connected by $\star$ and multiplied by $(r_j, \tilde{r}_i)$; (v) integrate over $\mathcal{M}$. Actually, this is exactly the procedure one uses in usual commutative theories modulo ordering ambiguities following from the noncommutativity.

The following Theorem demonstrates that the operation we have just defined gives indeed a Poisson structure on the space of star-local functionals.

Theorem 2.1 *Let R, $\tilde{R}$ and $\hat{R}$ be star-local functionals on the phase space. Then*
(1) $\{R,\tilde{R}\} = -\{\tilde{R},R\}$ *(antisymmetry),*
(2) $\{\{R,\tilde{R}\},\hat{R}\} + \{\{\hat{R},R\},\tilde{R}\} + \{\{\tilde{R},\hat{R}\},R\} = 0$ *(Jacobi identity).*

Proof. We start with noting that since we do not specify the origin of the canonical variables, the time coordinate does not play any significant role, and the statements above (almost) follow from the standard analysis [8]. However, it is instructive to present here a complete proof as it shows that one do not need to rewrite the star product through infinite series of derivatives (so that the $\star$ product indeed plays a role of multiplication). Again, it is enough to study the case when all functionals are monomial ones. Then the first assertion follows from (17) and $(r_j, r_k) = -(r_k, r_j)$. Let

$$\hat{R} = \int d^D x\, \partial_{\hat{\kappa}_1} \hat{r}_1 \star \partial_{\hat{\kappa}_2} \hat{r}_2 \star \dots \partial_{\hat{\kappa}_p} \hat{r}_p \,. \tag{18}$$

Consider $\{\{R,\tilde{R}\},\hat{R}\}$. The first of the brackets "uses up" an r_j and an $\tilde{r}_i$. The second bracket uses a variable with hat and another variable either from R or from $\tilde{R}$. Consider first the terms in the repeated bracket which use twice some variables from R. All such terms combine into the sum

$$\sum_{i,k,j\neq l} (-1)^{|\tilde{\kappa}_i| + |\hat{\kappa}_k|} (r_j, \tilde{r}_i)(r_l, \hat{r}_k) \int d^D x\, \partial_{\kappa_{l+1}} r_{l+1} \star \dots \partial_{\kappa_{j-1}} r_{j-1} \\ \star \partial_{\kappa_j + \tilde{\kappa}_i} \left(\partial_{\tilde{\kappa}_{i+1}} \tilde{r}_{i+1} \star \dots \partial_{\tilde{\kappa}_{i-1}} \tilde{r}_{i-1} \right) \star \partial_{\kappa_{j+1}} r_{j+1} \star \dots \partial_{\kappa_{l-1}} r_{l-1} \\ \star \partial_{\kappa_l + \hat{\kappa}_k} \left(\partial_{\hat{\kappa}_{k+1}} \hat{r}_{k+1} \star \dots \partial_{\hat{\kappa}_{k-1}} \hat{r}_{k-1} \right)$$

This complicated expression is symmetric with respect to interchanging the roles of the variables with hats and the variables with tilde. Therefore, it is clear that the terms having two brackets with r in $\{\{\hat{R},R\},\tilde{R}\}$ have exactly the same form as above but with a minus sign. No such terms (with two brackets with r) may appear in $\{\{\tilde{R},\hat{R}\},R\}$. Therefore, this kind of terms are cancelled in $\{\{R,\tilde{R}\},\hat{R}\}+\{\{\hat{R},R\},\tilde{R}\}+\{\{\tilde{R},\hat{R}\},R\}$. By repeating the same arguments for $\hat{r}$ and $\tilde{r}$ one proves our second assertion. □

One can define a canonical bracket between functionals and densities (star-local expressions) by the equation:

$$\{R,h(r)(x)\} := \frac{\delta}{\delta\beta(x)}\left\{R,\int d^Dy\,\beta(y)\star h(r)(y)\right\}. \tag{19}$$

To construct brackets between two densities (i.e., to give a proper extension of (6) to nonlinear functions) one has to define star-products with delta-functions which may be a very non-trivial task. We shall never use brackets between densities.

To use the canonical bracket in computation of variations we need the following technical Lemma.

Lemma 2.2 *Let p^a and q_b depend smoothly on a parameter τ. We assume that the variables $\alpha(x)$ (these are the ones which do not have canonical conjugates) do not depend on τ. Let $h(r(\tau))$ be a star-local expression on the phase space. Then*

$$\partial_\tau\int d^Dx\beta\star h(r(\tau)) = \int d^Dx\,\left((\partial_\tau q_a)\star\left\{\int d^Dy\beta\star h(r),p^a(x)\right\}\right.$$
$$\left.-(\partial_\tau p^a)\star\left\{\int d^Dy\beta\star h(r),q_a(x)\right\}\right) \tag{20}$$

Proof. Obviously, it is enough to prove this Lemma for $\beta=1$. Let us consider first the case when just one of the canonical variables (say, p^b for a just single value of b) depends on τ, and when $h(r)=h_1(r)\star\partial_\kappa p^b\star h_2(r)$ where neither h_1 nor h_2 depend on p^b. Then

$$\partial_\tau\int d^Dx\,h(r) = \int d^Dx\,h_1\star\partial_\kappa(\partial_\tau p^b)\star h_2 = (-1)^{|\kappa|}\int d^Dx\,\partial_\kappa(h_2\star h_1)\star\partial_\tau p^b\,. \tag{21}$$

On the other hand, by using (17), one obtains

$$\left\{\int d^Dx\,h(r),\int d^Dy\,\beta(y)\star q_b(y)\right\} = -(-1)^{|\kappa|}\int d^Dx\,\partial_\kappa(h_2\star h_1)\star\beta\,. \tag{22}$$

Next we use (19) to see that the statement of this Lemma is indeed true for the simplified case considered. In general case one has to sum up many individual

contributions to both sides of (20) from different canonical variables occupying various places in h. Each of this contributions can be treated in the same way as above. $\square$

As an application, consider a noncommutative field theory described by the action

$$S = \int (p^a \partial_t q_a - h(p, q, \lambda))\, d^D x = \int p^a \partial_t q_a d^D x - H\,, \tag{23}$$

where h is a star-local expression, it contains temporal derivatives only implicitly, i.e. only though the Moyal star. Note, that due to (3) the star between p^a and $\partial_t q_a$ can be omitted. If one takes into account explicit time derivatives only, one can write $p^a = \delta S/(\delta \partial_t q_a)$. Then, $H = S - \int p \partial_t q d^D x$.

The equations of motion generated from the action (23) by taking variations with respect to q and p can be written in the "canonical" form:

$$\partial_t p^a + \{H, p^a\} = 0\,, \qquad \partial_t q_a + \{H, q_a\} = 0 \tag{24}$$

This can be easily shown by taking $q(\tau) = q + \tau \delta q$ and $p(\tau) = p + \tau \delta p$ and using Lemma 2.2. No explicit time derivative acts on λ. In a commutative theory λ generates constraints.

3 Constraints and gauge symmetries

Let us specify the form of (23):

$$S = \int \left(p^a \partial_t q_a - \lambda^j \star G_j(p, q) - h(p, q)\right) d^D x \tag{25}$$

We shall call $G_j(p, q)$ a constraint, although due to the presence of the Moyal star it cannot be interpreted as a condition on a space-like surface. Dirac classification of the constraints can be also performed with the modified canonical bracket. We say that the constraints $G_j(p, q)$ are first-class if their brackets with $h(p, q)$ and between each other are again constraints, i.e.,

$$\left\{ \int d^D x \alpha^i \star G_i, \int d^D x \beta^j \star G_j \right\} = \int d^D x C(p, q; \alpha, \beta)^k \star G_k\,, \tag{26}$$

$$\left\{ \int d^D x \alpha^i \star G_i, \int d^D x h(p, q) \right\} = \int d^D x B(p, q; \alpha)^k \star G_k\,. \tag{27}$$

By Theorem 2.1(1) the structure functions are antisymmetric, $C(p, q; \alpha, \beta)^j = -C(p, q; \beta, \alpha)^j$. Further restrictions on C and B follow from the Jacobi identities (cf. Theorem 2.1(2)).

Theorem 3.1 *Let $G_i(p,q)$ be fist-class constraints (so that (26) and (27) are satisfied). Then the transformations*

$$\delta p^a = \left\{ \int d^D x \alpha^j \star G_j,\, p^a \right\} \tag{28}$$

$$\delta q_b = \left\{ \int d^D x \alpha^j \star G_j,\, q_b \right\} \tag{29}$$

$$\delta \lambda^j = -\partial_t \alpha^j - C(p,q;\alpha,\lambda)^j - B(p,q;\alpha)^j \tag{30}$$

with arbitrary α^j are gauge symmetries of the action (25).

Proof. To prove this Theorem we simply check invariance of (25) under (28) - (30). Let $f(p,q)$ be an arbitrary star-local expression depending on the canonical variables p and q only. Then, by (28) and (29),

$$\delta\, f(p,q) = \left\{ \int d^D x \alpha^j \star G_j,\, f(p,q) \right\} \tag{31}$$

It is now obvious that that the transformations of G and h in the action (25) are compensated by the second and third terms in $\delta\lambda$ respectively. The remaining term in the action transforms as

$$\begin{aligned}
&\delta \int d^D x\, p^a \partial_t q_a = \\
&= \int d^D x \left(\left\{ \int d^D y \alpha^j \star G_j, p^a(x) \right\} \star \partial_t q_a + p^a \star \partial_t \left\{ \int d^D y \alpha^j \star G_j, q_a(x) \right\} \right) \\
&= \int d^D x \left(\left\{ \int d^D y \alpha^j \star G_j, p^a(x) \right\} \star \partial_t q_a - (\partial_t p^a) \star \left\{ \int d^D y \alpha^j \star G_j, q_a(x) \right\} \right) \\
&= \int d^D x \alpha^j \star \partial_t G_j = -\int d^D x \partial_t(\alpha^j) \star G_j
\end{aligned} \tag{32}$$

Here we used integration by parts and Lemma 2.2. The last term in (32) is compensated by the first (gradient) term in the variation (30). Therefore, the action (25) is indeed invariant under (28) - (30).□

Let us compare the technique developed here to the Ostrogradski formalism for theories with higher time derivatives. In this formalism [14, 9] new phase space variables $P(t,T) = p(t+T)$ and $Q(t,T) = q(t+T)$ are introduced. Then t is interpreted as an evolution parameter, while T labels degrees of freedom (number of degrees of freedom is proportional to the order of temporal derivatives). Then a delta-function $\delta(T-T')$ appears naturally on the right hand side of the Poisson brackets between Q and P calculated at the same value of t. By returning (naively) to the original variables q and p one obtains

(6). In the approach of [9] one proceeds in a different way. The resulting dynamical system is interpreted as a system with an infinite number of second-class constraints. Additional first-class constraints would lead to considerable complications in this procedure. It may happen that these two approaches are equivalent, but this requires further studies.

4 Noncommutative gravity in two dimensions

In [24] we considered an example of a two-dimensional topological noncommutative gauge theory[4] which was equivalent to a noncommutative version [21] of the Jackiw-Teitelboim gravity [16]. It was the only noncommutative gravity in two dimensions known that far. In this section we construct a new model and analyse its' gauge symmetries.

Consider the action

$$S = \frac{1}{4} \int d^2x \, \varepsilon^{\mu\nu} \left[\phi_{ab} \star R^{ab}_{\mu\nu} - 2\varepsilon_{ab} \Lambda e^a_\mu \star e^b_\nu - 2\phi_a \star T^a_{\mu\nu} \right] \tag{33}$$

with the curvature tensor

$$\begin{aligned} R^{ab}_{\mu\nu} = \varepsilon^{ab} & \left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \frac{i}{2} [\omega_\mu, b_\nu] + \frac{i}{2} [b_\mu, \omega_\nu] \right) \\ & + \eta^{ab} \left(i\partial_\mu b_\nu - i\partial_\nu b_\mu + \frac{1}{2} [\omega_\mu, \omega_\nu] - \frac{1}{2} [b_\mu, b_\nu] \right) \end{aligned} \tag{34}$$

and with the noncommutative torsion

$$\begin{aligned} T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \frac{1}{2} \varepsilon^a{}_b \left([\omega_\mu, e^b_\nu]_+ - [\omega_\nu, e^b_\mu]_+ \right) \\ + \frac{i}{2} \left([b_\mu, e^a_\nu] - [b_\nu, e^a_\mu] \right) . \end{aligned} \tag{35}$$

The fields ϕ and ψ are combined into

$$\phi_{ab} := \phi \varepsilon_{ab} - i \eta_{ab} \psi \, . \tag{36}$$

Here $[\ ,\ ]_+$ denotes anticommutators. Both commutators and anticommutators are calculated with the Moyal star. Noncommutative curvature and torsion were derived in [21].

We use the tensor $\eta^{ab} = \eta_{ab} = \mathrm{diag}(+1, -1)$ to move indices up and down. The Levi-Civita tensor is defined by $\varepsilon^{01} = -1$, so that the following relations hold

$$\varepsilon^{10} = \varepsilon_{01} = 1, \qquad \varepsilon^0{}_1 = \varepsilon^1{}_0 = -\varepsilon_0{}^1 = -\varepsilon_1{}^0 = 1 \, . \tag{37}$$

[4]In a space-space noncommutative theory similar calculations were done in [25].

These relations are valid for both ε^{ab} and $\varepsilon^{\mu\nu}$. Note, that $\varepsilon^{\mu\nu}$ is always used with both indices up.

In the commutative limit the fields b_μ and ψ decouple, and the action becomes equivalent to (14) with $U = 0$ and $V = \tilde{V}$ given in (13).

An additional $U(1)$ gauge field is typically necessary to close the gauge algebra in NC case. This field may play also another role: by adding an additional abelian gauge field one can overcome the non-existence theorem of [26] for a dilaton action for the so-called exact string black hole [27] and construct a suitable action with the extended set of fields [28].

It is crucial to prove that the model (33) indeed has right number of gauge symmetries. According to our analysis it is enough to show that the constraint algebra closes w.r.t. to the bracket defined above. One can rewrite (33) in the canonical form:

$$S = \int d^2x \left(p^i \partial_0 q_i - \lambda^i \star G_i \right) \tag{38}$$

(cf. (25)). Here:

$$\begin{aligned} q_i &= (e_1^a, \omega_1, b_1), \\ p^i &= (\phi_a, \phi, -\psi), \\ \lambda^i &= (e_0^a, \omega_0, b_0). \end{aligned} \tag{39}$$

The constraints are

$$G_a = -\partial_1 \phi_a + \frac{1}{2}\varepsilon^b{}_a [\omega_1, \phi_b]_+ + \frac{i}{2}[\phi_a, b_1] - \Lambda \varepsilon_{ab} e_1^b \,, \tag{40}$$

$$G_3 = -\partial_1 \phi + \frac{i}{2}[\phi, b_1] + \frac{i}{2}[\psi, \omega_1] - \frac{1}{2}\varepsilon^a{}_b [\phi_a, e_1^b]_+ \,, \tag{41}$$

$$G_4 = \partial_1 \psi - \frac{i}{2}[\psi, b_1] + \frac{i}{2}[\phi, \omega_1] + \frac{i}{2}[\phi_a, e_1^a] \,. \tag{42}$$

The following formulae hold for arbitrary trace operation on an operator algebra. In our case, this trace is just a space-time integral.

$$\mathrm{Tr}([A_1, B_1][B_2, A_2] - [B_1, A_2][A_1, B_2]) = -\mathrm{Tr}([A_1, A_2][B_1, B_2]) \tag{43}$$

$$\mathrm{Tr}([A_1, B_1]_+[A_2, B_2]_+ - [A_1, B_2]_+[A_2, B_1]_+) = -\mathrm{Tr}([A_1, A_2][B_1, B_2]) \tag{44}$$

$$\mathrm{Tr}([A_1, B_1]_+[B_2, A_2] - [B_1, A_2]_+[A_1, B_2]) = \mathrm{Tr}([B_1, B_2][A_1, A_2]_+) \tag{45}$$

These formulae help to transform the brackets into a factorized form $\int C(\alpha, \beta)\star$

$G(p,q)$. The constraint algebra indeed closes and reads

$$\left\{\int \alpha^a \star G_a, \int \beta^b \star G_b\right\} = 0 \tag{46}$$

$$\left\{\int \alpha \star G_3, \int \beta \star G_3\right\} = \frac{i}{2}\int [\alpha,\beta] \star G_4 \tag{47}$$

$$\left\{\int \alpha \star G_4, \int \beta \star G_4\right\} = -\frac{i}{2}\int [\alpha,\beta] \star G_4 \tag{48}$$

$$\left\{\int \alpha \star G_3, \int \beta \star G_4\right\} = -\frac{i}{2}\int [\alpha,\beta] \star G_3 \tag{49}$$

$$\left\{\int \alpha \star G_3, \int \beta^a \star G_a\right\} = -\frac{1}{2}\int [\alpha,\beta^a]_+ \, \varepsilon^b{}_a \star G_b \tag{50}$$

$$\left\{\int \alpha \star G_4, \int \beta^a \star G_a\right\} = -\frac{i}{2}\int [\alpha,\beta^a] \star G_a \tag{51}$$

Here $\int := \int d^2x$.

One can easily find gauge symmetries of the action. The transformations generated by G_a read:

$$\begin{aligned}
&\delta e^a_\mu = -\partial_\mu \alpha^a - \frac{1}{2}\varepsilon^a{}_c[\omega_\mu,\alpha_c]_+ - \frac{i}{2}[b_1,\alpha^a], \\
&\delta\omega_\mu = \delta b_\mu = 0\,, \\
&\delta\phi = \frac{1}{2}\varepsilon^b{}_a[\alpha^a,\phi_b]_+\,, \qquad \delta\psi = -\frac{i}{2}[\alpha^a,\phi_a]\,, \\
&\delta\phi_a = -\Lambda\alpha^b\varepsilon_{ba}\,.
\end{aligned} \tag{52}$$

The constraint G_3 generates

$$\begin{aligned}
&\delta e^a_\mu = \frac{1}{2}\varepsilon^a{}_b[e^b_\mu,\beta]_+\,, \qquad \delta b_\mu = \frac{i}{2}[\omega_\mu,\beta]\,, \\
&\delta\omega_\mu = -\partial_\mu\beta - \frac{i}{2}[b_\mu,\beta]\,, \\
&\delta\phi_a = -\frac{1}{2}\varepsilon^c{}_a[\beta,\phi_c]_+, \qquad \delta\phi = \frac{1}{2}[\beta,\psi]\,, \qquad \delta\psi = -\frac{i}{2}[\beta,\phi]\,.
\end{aligned} \tag{53}$$

Gauge symmetries generated by G_4 are:

$$\begin{aligned}
&\delta e^a_\mu = -\frac{i}{2}[e^a_\mu,\gamma]\,, \qquad \delta\omega_\mu = -\frac{i}{2}[\omega_\mu,\gamma]\,, \\
&\delta b_\mu = -\partial_\mu\gamma - \frac{i}{2}[b_\mu,\gamma]\,, \\
&\delta\phi_a = \frac{i}{2}[\gamma,\phi_a]\,, \qquad \delta\phi = \frac{i}{2}[\gamma,\phi]\,, \qquad \delta\psi = \frac{i}{2}[\gamma,\psi]\,.
\end{aligned} \tag{54}$$

In (52) - (54) the functions α^a, β and γ denote parameters of the gauge transformations.

In the commutative limit the transformations (52) and (53) become equivalent to diffeomorphisms and local Lorentz transformations up to a field-dependent redefinition of the parameters (the symmetry (54) decouples completely). Therefore, we may say that gauge symmetries of the noncommutative action (33) contain noncommutative deformations of Lorentz and diffeomorphism group. This is a rather nontrivial fact since $\theta_{\mu\nu}$ remains constant under the transformations. A more elaborate discussion on noncommutative diffeomorphism in two dimension can be found in [21]. Unfortunately, it is not clear so far how one may construct a gauge invariant line element.

To deform the Witten black hole one may use its formulation as a Wess-Zumino-Novikov-Witten (WZWN) theory. A noncommutative formulation of the $U(2)/U(1)$ WZWN model was constructed in [29]. The paper [29] does not analyse gravity aspects of the model. It remains unclear whether the deformation of [29] is equivalent to the one presented above. The action (33) may also be obtained as a singular limit of the noncommutative JT model [21]. To prove that the gauge symmetries are preserved in this limit is of the same level of complexity as the direct analysis presented above.

Somewhat surprisingly, construction of a proper noncommutative deformation of classical action having proper number of gauge symmetries is the hardest part of the job. Analysing classical solution seems to be rather straightforward. Indeed, let us impose the gauge condition

$$e_0^+ = 0, \quad e_0^- = 1, \quad \omega_0 = 0, \quad b_0 = 0, \tag{55}$$

where $e_\mu^\pm = 2^{-1/2}\left(e_\mu^0 \pm e_\mu^1\right)$. Then, as one can easily see, the equations of motion become linear and the model can be solved in a rather straightforward way. Therefore, classical analysis of the noncommutative model considered here is similar to what we have in the commutative case (see [15] for more details). However, transition between different formulations of the dilaton gravities remains a problem. For example, it is not clear how one should generalise the dilaton-dependent conformal transformation described in sec. 1 to the noncommutative case.

The gauge condition (55) is the main technical ingredient of exact path integral quantisation of two-dimensional commutative dilaton gravities [20]. In the case of noncommutative JT model this gauge condition also allowed to calculate the path integral exactly [22]. Adding the matter fields to this formalism [30] may cause a problem.

Let us conclude this section with some remarks on possibility of fixed background perturbative calculations in noncommutative gravity theories. At least at one-loop order the heat kernel technique [31] seems to be an adequate tool.

A generalisation of the heat kernel expansion on flat Moyal spaces was constructed recently [32]. Even on curved Moyal manifolds one can calculate leading heat kernel coefficients and construct a generalisation of the Polyakov action [22]. It is crucial that the operator describing quantum fluctuations contains only left or only right star multiplications. If both types of multiplications are present simultaneously, the heat kernel expansion seems to be modified in an essential way [33] due to the mixing of ultra-violet and infra-red scales discovered previously in Feynman diagrams [34].

5 Conclusions

In this paper we have suggested a modification of the Poisson bracket which is defined on fields at different values of the time coordinate. In this modified canonical formalism, only explicit time derivatives (i.e., the ones which are not hidden in the Moyal multiplication) define the canonical structure. Although this means serious deviations from standard canonical methods, the resulting brackets still satisfy the Jacobi identities (Theorem 2.1) and generate classical equations of motion. Our main result (Theorem 3.1) is that we can still define the notion of fist-class constraints, which generate gauge symmetries, and these symmetries are written down explicitly[5]. It would be interesting to construct a classical BRST formalism starting with our brackets. Anyway, it is important to restore the reputation of space-time noncommutative theories. This is required by the principles of symmetry between space and time, but also by interesting physical phenomena which appear due to the space time noncommutativity (just as an example we may mention creation of bound states with hadron-like spectra [36]). To avoid confusions we stress that our analysis is purely classical. It is not clear whether our brackets can be used for quantisation at all.

As an application of the canonical formalism we considered noncommutative gravity and constructed a new deformed dilaton gravity in two dimensions (which is a conformally transformed string gravity). Naively one would expect that the presence of constant $\theta_{\mu\nu}$ destrois a part of the symmetries (and this really happens in non-gravitational noncommutative theories). In our case, however, we observe just right number of gauge symmetries in the deformed theory. It seems that noncommutativity naturally leads to gravity, as well as gravity naturally leads to noncommutativity [2].

[5] Just existence of the symmetries does not come as a great surprise in the view of the analysis of [35] which is valid for theories with arbitrary (but finite!) order of time derivatives. An important feature of the present approach is rather simple explicit formulae similar to that in the case of commutative theories with 1st order time derivatives.

Acknowledgements

This paper is based on lectures given at the Advanced Summer School on Modern Mathematical Physics (Dubna) and the International Fock School (St.Petersburg). I am grateful to A. T. Filippov, V. Yu. Novozhilov and Yu. V. Novozhilov for their kind hospitality. I benefited from numerous discussions with Daniel Grumiller. This work was supported in part by the DFG Project BO 1112/12-2 and by the Multilateral Research Project "Quantum gravity, cosmology and categorification" of the Austrian Academy of Sciences and the National Academy of Sciences of the Ukraine.

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